

STRONG TIME-PERIODIC SOLUTIONS TO THE 3D PRIMITIVE EQUATIONS SUBJECT TO ARBITRARY LARGE FORCES

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ABSTRACT. We show that the three-dimensional primitive equations admit a strong time-periodic solution of period $T > 0$, provided the forcing term $f \in L^2(0, T; L^2(\Omega))$ is a time-periodic function of the same period. No restriction on the magnitude of f is assumed. As a corollary, if, in particular, f is time-independent, the corresponding solution is steady-state.

1. INTRODUCTION

Consider the primitive equations in the isothermal setting, i.e. assuming that the temperature θ equals a constant θ_0 . In this case, the primitive equations consist of the following set of equations

$$(1.1) \quad \begin{aligned} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi &= f & \text{in } \Omega \times (0, T), \\ \partial_z \pi &= 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} u &= 0 & \text{in } \Omega \times (0, T), \\ v(0) &= a. \end{aligned}$$

Here $\Omega = G \times (-h, 0)$, where $G = (0, 1)^2$, $h > 0$, and $T > 0$. The velocity u of the fluid is given by $u = (v, w)$ with $v = (v_1, v_2)$, and where v and w denote the horizontal and vertical components of u , respectively. Furthermore, π denotes the pressure of the fluid (more precisely, $\pi = p + \theta_0 z$, where p is the original pressure, $z \in (-h, 0)$) and f a given external force. The symbol $\nabla_H = (\partial_x, \partial_y)^\top$ denotes the horizontal gradient, Δ the three-dimensional Laplacian and ∇ and div the three dimensional gradient and divergence operators. The above equations take into account, by scale analysis, the hydrostatic approximation of the Navier-Stokes equations; for more details see e.g. [16], [17].

The system is complemented by the set of boundary conditions

$$(1.2) \quad \begin{aligned} \partial_z v &= 0, \quad w = 0 & \text{on } \Gamma_u \times (0, T), \\ v &= 0, \quad w = 0 & \text{on } \Gamma_b \times (0, T), \\ u, \pi &\text{ are periodic} & \text{on } \Gamma_l \times (0, T). \end{aligned}$$

Here $\Gamma_u := G \times \{0\}$, $\Gamma_b := G \times \{-h\}$, $\Gamma_l := \partial G \times [-h, 0]$ denote the upper, bottom and lateral parts of the boundary $\partial\Omega$, respectively.

The full primitive equations were introduced and investigated for the first time by Lions, Temam and Wang in [10, 11]. They proved the existence of a global weak solution for this set of equations for initial data $a \in L^2$. The existence of a local, strong solution with data $a \in H^1$ was proved first by Guillén-González, Masmoudi and Rodríguez-Bellido in [4].

In 2007, Cao and Titi [1] proved a breakthrough result for this set of equations which states, roughly speaking, that there exists a unique, *global strong* solution to the primitive equations for *arbitrary* initial data $a \in H^1$. Note that the boundary conditions on $\Gamma_b \cup \Gamma_l$ considered there are different from the ones we are imposing in (1.2). Successively, in [8] Kukavica and Ziane considered the primitive equations subject to boundary conditions as in (1.2), and proved global strong well-posedness of the

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primitive equations with respect to arbitrary H^1 -data. For different approaches see also Kobelkov [7] and Kukavica, Pei, Rusin and Ziane [9].

It is worth emphasizing that, while the fundamental problem of global existence and uniqueness for the initial-value problem can be considered to a great extent settled, at least in the L^2 framework, other important issues like existence of *strong* time-periodic (and, in particular, steady-state) solutions to the primitive equations appear to be at a stage where further investigation is still required. In this regard, we recall that the question of whether system (1.1) admits time-periodic solutions was first addressed by Tachim Medjo [15].¹ There, under the assumption of “*small*” *forcing term*, existence (and uniqueness) of strong solutions is achieved by the classical Galerkin method. More recently, Hsia and Shiue [6] proved a similar result by a different method suggested by Serrin [14], again under a suitable *smallness condition* on the forcing term. Furthermore, they showed asymptotic stability of such solutions when the initial perturbations are sufficiently small. Notice that, as corollary, both results in [15, 6] furnish existence of steady-state solutions to the primitive equations for forcing terms of *restricted* magnitude.

At this point, it must be observed that the smallness assumption on the forcing term is undesired and, most of all, appears somehow unexpected if one compares this situation with the classical Navier-Stokes theory. There, even though the initial-value problem still lacks of a global existence result for strong solutions with initial data of arbitrary size, nevertheless the steady-state boundary-value problem is known to have a smooth solution for (smooth) forcing term of arbitrary magnitude since the fundamental work of J.Leray. However, it should be added, also in the light of the contributions [15, 6], that the achievement of a result of this type for the primitive equations does not seem to be obvious, at least if one uses the classical methods employed for the Navier-Stokes equations.

The main objective of this article is to prove existence of *strong* time-periodic solutions to the primitive equations of the form (1.1)_{1,2,3} for *arbitrary* (time-periodic) $f \in L^2(0, \mathcal{T}, L^2(\Omega))$, hereby *without assuming any smallness condition on f* . As a byproduct, this result provides an analogous one for steady-state solutions.

As we hinted earlier on, the approach we use differs from “standard” ones, and is based on the following three steps: First, we construct a (suitable) *weak* time-periodic solution, v , to (1.1)_{1,2,3} corresponding to the given f , by combining classical Galerkin’s method with Brouwer’s fixed point theorem. Secondly, we show the existence of a unique, *strong* solution u to the *initial-value* problem (1.1) for arbitrary $f \in L^2(0; \mathcal{T}; L^2(\Omega))$, and a in a subspace of $H^1(\Omega)$, by using the arguments of [5]. Finally, we look at v as a weak solution to the *initial-value* problem and employ a weak-strong uniqueness result of the type proved by Guillén-González, Masmoudi and Rodríguez-Bellido in [4], which then implies $v \equiv u$, thus furnishing the main results of this article stated as Theorem 2.4 and Corollary 2.5.

The plan of the paper is the following. In Section 2 we make some preliminary considerations and give the statement of our main results (Theorem 2.4 and Corollary 2.5). In Section 3 we then show the existence of a weak time-periodic solution corresponding to forcing terms of arbitrary size. The following Section 4 is dedicated to the proof of existence and uniqueness of (an equivalent form of) the initial-value problem (1.1) for arbitrary f and a in appropriate function classes. Finally, in Section 5, we give the proof of Theorem 2.4.

2. PRELIMINARIES AND MAIN RESULTS

Following Lions, Temam and Wang [10, 11] and Cao and Titi [1], we rewrite the primitive equations given in (1.1) subject to the boundary conditions (1.2) in the following equivalent form. Since the vertical component w of u is determined by the incompressibility condition we have

$$w(x, y, z) = \int_z^0 \operatorname{div}_H v(x, y, \zeta) d\zeta, \quad (x, y) \in G, \quad -h < z < 0,$$

¹Even in the more general non-isothermal framework.

due to the boundary condition $w = 0$ on Γ_u . The further boundary condition $w = 0$ on Γ_b gives rise to the constraint

$$\operatorname{div}_H \bar{v} = 0 \quad \text{in } G,$$

where \bar{v} stands for the average of v in the vertical direction, i.e.,

$$(2.1) \quad \bar{v}(x, y) := \frac{1}{h} \int_{-h}^0 v(x, y, z) dz, \quad (x, y) \in G.$$

Then problem (1.1)-(1.2) is equivalent to finding a function $v : \Omega \rightarrow \mathbb{R}^2$ and a function $\pi : G \rightarrow \mathbb{R}$ satisfying the set of equations

$$(2.2) \quad \begin{aligned} \partial_t v + v \cdot \nabla_H v + w(v) \partial_z v - \Delta v + \nabla_H \pi &= f && \text{in } \Omega \times (0, T), \\ w(v) &= \int_z^0 \operatorname{div}_H v d\zeta && \text{in } \Omega \times (0, T), \\ \operatorname{div}_H \bar{v} &= 0 && \text{in } G \times (0, T), \\ v(0) &= a, \end{aligned}$$

as well as the boundary conditions

$$(2.3) \quad \begin{aligned} \partial_z v &= 0 && \text{on } \Gamma_u \times (0, T), \\ v &= 0 && \text{on } \Gamma_b \times (0, T), \\ v \text{ and } \pi &\text{ are periodic} && \text{on } \Gamma_l \times (0, T). \end{aligned}$$

The following terminology for describing the periodic boundary conditions will be useful. Let $m \in \{0, 1\}$. We then say that a smooth function $f : \bar{\Omega} \rightarrow \mathbb{R}$ is *space periodic of order m on Γ_l* if

$$\frac{\partial^\alpha f}{\partial x^\alpha}(0, y, z) = \frac{\partial^\alpha f}{\partial x^\alpha}(1, y, z) \quad \text{and} \quad \frac{\partial^\alpha f}{\partial y^\alpha}(x, 0, z) = \frac{\partial^\alpha f}{\partial y^\alpha}(x, 1, z),$$

for all $\alpha = 0, \dots, m$. Note that we do not consider any symmetry conditions in the z -direction. The Sobolev spaces equipped with space-periodic boundary conditions in the horizontal directions are defined by²

$$H_{\text{per}}^m(\Omega) := \{f \in H^m(\Omega) : f \text{ is space-periodic of order } m-1 \text{ on } \Gamma_l\}.$$

Note that $C_{\text{per}}^\infty(\bar{\Omega}) := \{f \in C^\infty(\bar{\Omega}) : f \text{ is space-periodic of arbitrary order on } \Gamma_l\}$ is dense in $H_{\text{per}}^m(\Omega)$. We now introduce the function spaces \mathbb{H}, \mathbb{H}^1 and \mathbb{H}^2 by

$$\mathbb{H}(\Omega) := \{v \in L_{\text{per}}^2(\Omega) : \operatorname{div}_H \bar{v} = 0\}$$

$$\mathbb{H}^1(\Omega) := \{v \in H_{\text{per}}^1(\Omega) : \operatorname{div}_H \bar{v} = 0, v = 0 \text{ on } \Gamma_b\}$$

$$\mathbb{H}^2(\Omega) := \{v \in \mathbb{H}^1(\Omega) \cap H^2(\Omega)_{\text{per}} : \partial_z v = 0 \text{ on } \Gamma_u\}.$$

and denote its norms by $\|\cdot\|_2, \|\cdot\|_{\mathbb{H}^1}, \|\cdot\|_{\mathbb{H}^2}$, respectively. As usual, (\cdot, \cdot) stands for the usual L^2 scalar product.

Moreover, given an interval $I \subset \mathbb{R}$, the space $C_w(I; \mathbb{H}(\Omega))$ stands for the class of functions $v : I \rightarrow \mathbb{H}(\Omega)$ such that $t \mapsto (v(t), \psi)$ is continuous for all $\psi \in \mathbb{H}(\Omega)$.

Finally, we say that a function $f \in L^1(0, T; L^1(\Omega))$, all $T > 0$, is T -periodic ($T > 0$) if $f(t, x) = f(t + T, x)$ for a.a. $(t, x) \in [0, \infty) \times \Omega$.

Definition 2.1. Let $T > 0$ and $f \in L^1(0, T; L^2(\Omega))$, all $T > 0$. A function $v : [0, \infty) \times \Omega \rightarrow \mathbb{R}^2$ is called *weak T -periodic solution* to (2.2)_{1,2,3} and (2.3) if

$$\text{i) } v \in C_w([0, T]; \mathbb{H}(\Omega)) \cap L^2(0, T; \mathbb{H}^1(\Omega)) \text{ for all } T > 0,$$

²Throughout the paper we shall use the same font style to denote scalar, vector and tensor-valued functions and corresponding function spaces.

ii) For all $\mathcal{T} > 0$ and all $\varphi \in C^1([0, \mathcal{T}]; \mathbb{H}^1(\Omega)) \cap L^2(0, \mathcal{T}; \mathbb{H}^2(\Omega))$

$$\int_0^t \{ (v, \partial_t \varphi) - (\nabla v, \nabla \varphi) + (v \cdot \nabla_H \varphi, v + w(v) \partial_z \varphi, v) \} = - \int_0^t (f, \varphi) + (v(t), \varphi(t)) - (v(0), \varphi(0)), \quad t \in (0, \mathcal{T}),$$

iii) For all $\mathcal{T} > 0$, all $t \in (0, \mathcal{T}]$ and a.a. $s \in [0, t)$, v satisfies the *strong energy inequality*

$$\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2 + 2 \int_s^t (f(\tau), v(\tau)) d\tau,$$

iv) $v(t+T, x) = v(T, x)$ for all $t \geq 0$ and a.a. $x \in \Omega$.

A weak T -periodic solution v is called a *strong* if, in addition to i)–iv), it holds $v \in C([0, \mathcal{T}]; \mathbb{H}^1(\Omega)) \cap L^2(0, \mathcal{T}; \mathbb{H}^2(\Omega))$, $\partial_t v \in L^1(0, \mathcal{T}; L^2(\Omega))$, for all $\mathcal{T} > 0$.

If, in particular, $f \in \mathbb{H}(\Omega)$ is independent of $t \geq 0$, we say that $v_s \in \mathbb{H}^1(\Omega)$ is a *weak steady-state solution* to (2.2)_{1,2,3}–(2.3) if v_s satisfies condition ii) for all $\varphi \in \mathbb{H}^2(\Omega)$. A weak steady-state solution is called *strong* if $v_s \in \mathbb{H}^2(\Omega)$.

Remark 2.2. It is worth noticing that every term in ii) is well defined. This is obvious for all linear terms in v . As for the nonlinear ones, by Hölder's inequality and Sobolev embeddings we obtain for all $v_1, v_2 \in H^1(\Omega)$ and all $v_3 \in H^2(\Omega)$

$$(2.4) \quad \int_0^t |(v_1 \cdot \nabla_H v_3, v_2)| \leq \int_0^t \|v_1\|_3 \|\nabla_H v_3\|_2 \|v_2\|_6 \leq C \int_0^t \|v_1\|_2^{\frac{1}{2}} \|v_1\|_{H^1}^{\frac{1}{2}} \|v_2\|_{H^1} \|\nabla_H v_3\|_2.$$

Furthermore, using inequality (93) from [1], we show that

$$(2.5) \quad \int_0^t |(w(v_1) \partial_z v_3, v_2)| \leq C \int_0^t \|\nabla_H v_1\|_2 \|v_2\|_2^{\frac{1}{2}} \|v_2\|_{H^1}^{\frac{1}{2}} \|\partial_z v_3\|_2^{\frac{1}{2}} \|\partial_z v_3\|_{H^1}^{\frac{1}{2}},$$

which proves the claim. We note that the proof of the latter does not use boundary conditions for v_i , $i = 1, 2, 3$, which in [1] are different than those adopted here.

Remark 2.3. The above definition of a weak time-periodic solution is somewhat different than the one typically found in the literature. However, this formulation is needed when we will compare these solutions with solutions to the initial-value problem; see Section 4.

We are now in the position to state the main result of this article.

Theorem 2.4. *Let $T > 0$ and let $f \in L^2(0, T; L^2(\Omega))$, all $\mathcal{T} > 0$, be T -periodic. Then problem (2.2)_{1,2,3}–(2.3) has at least one corresponding strong T -periodic solution.*

The above result at once implies the following one.

Corollary 2.5. *Let $f \in L^2(\Omega)$. Then problem (2.2)_{1,2,3}–(2.3) has at least one corresponding strong steady-state solution.*

We emphasize at this point that, in contrast to previous known results [6, 15], our findings *do not* require any smallness condition on f .

3. WEAK TIME-PERIODIC SOLUTIONS

Objective of this section is to show that the class of weak T -periodic solutions to (2.2)_{1,2,3}–(2.3) is not empty under suitable assumptions on f . Precisely, we have the following.

Proposition 3.1. *Let $T > 0$, and let $f \in L^2(0, T; L^2(\Omega))$, all $\mathcal{T} > 0$, T -periodic. Then, there exists at least one weak T -periodic solution to (2.2)_{1,2,3}–(2.3)*

Proof. Even though our definition of a weak time-periodic solution is somehow different than the one usually given in the literature, the proof of its existence, based on the Faedo-Galerkin method, is quite standard; see, e.g., [13], [12, Chapter 4, §6.2], [3, pp. 256-260], [2]. For this reason, we shall only give the main arguments, referring the reader to the above papers for further details.

Let $\{\psi_n\} \subset \mathbb{H}^2(\Omega)$ be an orthonormal basis of $\mathbb{H}(\Omega)$ dense in $\mathbb{H}^1(\Omega)$ and $\mathbb{H}^2(\Omega)$. For example, we may take the eigenvectors of the hydrostatic Stokes operator in the L^2 setting (see [5, §4]). Let

$$v_m := \sum_{k=1}^m c_{mk}(t) \psi_k(x)$$

where, for all $r = 1, \dots, m$,

$$(3.1) \quad \frac{d}{dt}(v_m, \psi_r) = (v_m \cdot \nabla_H \psi_r, v_m) + (w(v_m) \partial_z \psi_r, v_m) + (\nabla v_m, \nabla \psi_r) + (f, \psi_r).$$

We now show a uniform bound in time on the functions $c_{mk}(t)$ for $k = 1, \dots, m$, which implies that the system (3.1) has a solution $c(t) := (c_{m1}(t), \dots, c_{mm}(t))$ for all $t \geq 0$ and all $m \geq 1$. In fact, multiplying both sides of (3.1) by $c_{mr}(t)$, summing over r , and taking into account that by (2.2)_{2,3,4}

$$(v_m \cdot \nabla_H v_m, v_m) + (w(v_m) \partial_z v_m, v_m) = 0,$$

we get

$$(3.2) \quad \frac{d}{dt} \|v_m\|_2^2 + 2 \|\nabla v_m(t)\|_2^2 = 2(f, v_m).$$

Since $v_m = 0$ at $x_3 = -h$, we have

$$(3.3) \quad \|v_m\|_2^2 \leq h^2 \|\partial_z v_m\|_2^2,$$

so that by Schwartz's inequality and (3.2) we infer

$$\frac{d}{dt} \|v_m\|_2 + \frac{2}{h^2} \|v_m\|_2 \leq 2\|f\|_2.$$

Integrating the latter from $t = 0$ to arbitrary $t > 0$ we get

$$(3.4) \quad e^{2t/h^2} \|v_m(t)\|_2 \leq \|v_m(0)\|_2 + 2 \int_0^t e^{2\tau/h^2} \|f(\tau)\|_2 d\tau,$$

thus deducing the claimed uniform bound for $c(t)$, once we observe that, by the orthonormality property of $\{\psi_n\}$, $|c(t)| = \|v_m(t)\|_2$.

Next, choose $R > 0$ such that $R(e^{2t/h^2} - 1) \geq 2 \int_0^T e^{2\tau/h^2} \|f(\tau)\|_2 d\tau$ and let \mathbb{B}_R^m the ball in \mathbb{R}^m centered at the origin with radius R . It follows from (3.4) that $|c(T)| = \|v_m(T)\|_2 \leq R$ provided $|c(0)| = \|v_m(0)\|_2 \leq R$. Thus the map

$$S : \mathbb{R}^m \ni c(0) \mapsto c(T) \in \mathbb{R}^M$$

maps \mathbb{B}_R^m into itself. Since it is also continuous, we conclude by Brouwer's theorem that for each $m \geq 1$ there exists $v_m(0)$ such that $v_m(0) = v_m(T)$. We may then extend $v_m(t)$ to the half-line $[0, \infty)$ to a periodic function of period T . Clearly, by (3.4),

$$(3.5) \quad \|v_m(t)\|_2 \leq R + 2 \int_0^t e^{2\tau/h^2} \|f(\tau)\|_2 d\tau, \quad t \geq 0.$$

Moreover, from (3.2), (3.3) and the time-periodicity of $v_m(t)$ we see that

$$(3.6) \quad \int_0^t \|\nabla v_m(\tau)\|_2^2 d\tau \leq h^4 \int_0^{\ell T} \|f(\tau)\|_2^2 d\tau, \quad t \in (0, \ell T), \ell > 0.$$

Furthermore, integrating both sides of (3.1) between 0 and t and using (2.4), (2.5), (3.5) and (3.6) we show that, for each fixed r , the sequence of functions $\{(v_m(t), \psi_r)\}$ is uniformly continuous and uniformly bounded. Combining this information with (3.1), (3.5)–(3.6), using (2.4)–(2.5), and following

the classical procedure (see, e.g., [2, pp. 18–20]) we show the existence of a function $v \in L^\infty(0, \mathcal{T}; \mathbb{H}(\Omega)) \cap L^2(0, \mathcal{T}; \mathbb{H}^1(\Omega))$ for all $\mathcal{T} > 0$ and a subsequence $\{v_{m'}\}$ such that for all $\mathcal{T} > 0$

$$(3.7) \quad \begin{aligned} v_{m'} &\rightarrow v && \text{weakly in } L^2(0, \mathcal{T}; \mathbb{H}^1(\Omega)), \\ v_{m'}(t) &\rightarrow v(t) && \text{weakly in } \mathbb{H}(\Omega) \text{ for all } t \in [0, \mathcal{T}], \\ v_{m'} &\rightarrow v && \text{strongly in } L^2(0, \mathcal{T}; \mathbb{H}(\Omega)). \end{aligned}$$

Recalling that $v_m(t+T) = v_m(T)$ for all $t \geq 0$, the second relation in (3.7) implies that v satisfies both properties i) and iv) of weak solutions. Furthermore, again by (3.7) and (3.2), we see that v satisfies also property iii). Finally, we integrate (3.1) over $(0, t)$ and then pass to the limit $(m') \rightarrow \infty$. Using (3.7) and (2.4)–(2.5) we see that

$$(v(t) - v(0), \psi_r) = \int_0^t (v \cdot \nabla_H \psi_r, v) + (w(v) \partial_z \psi_r, v) + (\nabla v, \nabla \psi_r) + (f, \psi_r),$$

for all $r \geq 1$. From this equation, taking into account the mentioned properties of $\{\psi_n\}$, again by classical arguments (e.g., [2, §2]) we prove that v satisfies also property ii), which concludes the proof. \square

4. EXISTENCE OF GLOBAL STRONG SOLUTIONS TO THE INITIAL-VALUE PROBLEM: INHOMOGENEOUS CASE

As mentioned earlier on, the basic idea for the proof of Theorem 2.4 is to compare the weak T -periodic solution of Proposition 3.1 with a strong global solution (u, p) to the inhomogeneous *initial-value problem* (2.2)–(2.3), for an appropriate choice of the initial data a . As customary [4, 1], by “strong” we mean $u \in C([0, \mathcal{T}]; \mathbb{H}^1(\Omega)) \cap L^2(0, \mathcal{T}; \mathbb{H}^2(\Omega))$ with $\partial_t u \in L^1(0, \mathcal{T}; \mathbb{H}(\Omega))$, and $p \in L^1(0, \mathcal{T}; H^1(\Omega))$, $\mathcal{T} > 0$. In [5], existence of such solutions has been established when $f \equiv 0$. In the following proposition, we shall suitably adapt the arguments of [5] to prove existence of global strong solutions to (2.2)–(2.3), when $f \not\equiv 0$ is prescribed in a proper function class.

Proposition 4.1. *Let $\mathcal{T} > 0$ arbitrary, $a \in \mathbb{H}^1(\Omega)$ and $f \in L^2(0, \mathcal{T}; L^2(\Omega))$. Then, problem (2.2)–(2.3) has a unique strong solutions in the interval $(0, \mathcal{T})$.*

Proof. The existence of a unique strong local solution was already proved in [4, Theorem 1.2]. Hence, in order to prove the assertion, it suffices to show that the velocity field u , of a given local, strong solution to the above problem, admits an a priori bound in the space $C([0, \mathcal{T}]; \mathbb{H}^1(\Omega)) \cap L^2(0, \mathcal{T}; \mathbb{H}^2(\Omega))$. Observe that this will also imply the stated properties on $\partial_t u$ and p , since from (2.2)₁ we first readily infer

$$\|\partial_t u\|_2 \leq \|u\|_\infty \|\nabla u\|_2 + \|\nabla u\|_4^2 + \|\Delta u\|_2, \quad \|\nabla_H p\|_2 \leq \|\partial_t u\|_2 + \|u\|_\infty \|\nabla u\|_2 + \|\nabla u\|_4^2 + \|\Delta u\|_2,$$

and then use the embedding $H^2(\Omega) \subset W^{1,4}(\Omega) \subset L^\infty(\Omega)$.

Noticing that $\|u\|_{H^2} \leq C\|\Delta u\|_2$, in order to show the above bound, it suffices to prove that

$$(4.1) \quad \|u(t)\|_{H^1}^2 + \int_0^t \|\Delta u(\tau)\|_2^2 d\tau \leq B(\|a\|_{\mathbb{H}^1}, \|f\|_{L^2_\tau(L^2)}, \mathcal{T}), \quad t \in [0, \mathcal{T}],$$

where $\|f\|_{L^2_\tau(L^2)} := (\int_0^\mathcal{T} \|f(\tau)\|_2^2 d\tau)^{1/2}$ and B is a continuous function. Here and hereafter, C denotes a generic constant.

In what follows, we shall closely employ the strategy of [5] and show how the main estimates obtained there in (6.5), (6.7), (6.9) and (6.10) modify if $f \not\equiv 0$ satisfies the stated assumptions. This will be achieved in *Steps 1–4* below, which will then lead to the proof of (4.1) in *Step 5*. We begin to observe that multiplying both sides of (2.1) (written for (u, p)) by u , integrating by parts and using Schwartz inequality and Poincaré inequality (3.3) we show

$$(4.2) \quad \|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \leq \|a\|_2^2 + h^2 \|f\|_{L^2_\tau(L^2)}^2, \quad t \in [0, \mathcal{T}].$$

The functions $\bar{u} = \frac{1}{h} \int_{-h}^0 u \, dz$ and $\tilde{u} := u - \bar{u}$ fulfill the following equations:

$$(4.3) \quad \begin{aligned} \partial_t \bar{u} - \Delta_H \bar{u} + \nabla_H p &= \bar{f} - \bar{u} \cdot \nabla \bar{u} - \frac{1}{h} \int_{-h}^0 (\tilde{u} \cdot \nabla_H \tilde{u} + \operatorname{div}_H u \tilde{u}) \, dz - \frac{1}{h} u_z|_{\Gamma_b} & \text{in } G, \\ \operatorname{div}_H \bar{u} &= 0 & \text{in } G, \end{aligned}$$

with $u_z := \partial_z u$, as well as

$$(4.4) \quad \partial_t \tilde{u} - \Delta \tilde{u} + \tilde{u} \cdot \nabla_H \tilde{u} + u_3 u_z + \bar{u} \cdot \nabla_H \tilde{u} = \tilde{f} - \tilde{u} \cdot \nabla_H \bar{u} + \frac{1}{h} \int_{-h}^0 (\tilde{u} \cdot \nabla_H \tilde{u} + \operatorname{div}_H u \tilde{u}) \, dz + \frac{1}{h} u_z|_{\Gamma_b} \quad \text{in } \Omega.$$

Step 1: Equation (4.3) implies

$$\begin{aligned} \partial_t \|\nabla_H \bar{u}(t)\|_{L^2(G)}^2 + \|\Delta_H \bar{u}\|_{L^2(G)}^2 + \|\nabla_H p\|_{L^2(G)}^2 \\ \leq C_1 (\|\bar{u}\|_{L^2(G)}^2 + \|\tilde{u}\|_{L^2(G)}^2 + \|u_z\|_{L^2(\Gamma_b)}^2 + \|f\|_2^2). \end{aligned}$$

Therefore, if $f \neq 0$ estimate (6.5) in [5] is replaced by

$$\begin{aligned} \partial_t \|\nabla_H \bar{u}\|_{L^2(G)}^2 + \|\nabla_H p\|_{L^2(G)}^2 &\leq C(\|u\|_2 + \|u\|_2^2)(\|u\|_{H^1} + \|u\|_{H^1}^2) \|\nabla_H \bar{u}\|_{L^2(G)}^2 + C_1 \|\tilde{u}\|_{L^2(G)}^2 \\ &\quad + \frac{1}{4} \|\nabla u_z\|_{L^2(G)}^2 + C(1 + \|u\|_2^2 + \|u\|_2^4) \|u\|_{H^1}^2 + C\|f\|_2^2. \end{aligned}$$

Step 2: Multiplying (2.2) by $-\partial_z u_z$ and integrating by parts leads to

$$\frac{1}{2} \partial_t \|u_z\|_2^2 + \|\nabla u_z\|_{L^2(\Omega)}^2 = - \int_G \nabla_H p \cdot u_z|_{\Gamma_b} - \int_\Omega (u_z \cdot \nabla_H u) \cdot u_z + \int_\Omega \operatorname{div}_H u \, u_z \cdot u_z - \int_\Omega f \cdot \partial_z u_z.$$

Therefore, by using Cauchy-Schwartz inequality in the latter and proceeding as in [5], we obtain that equation (6.7) in [5] generalizes to the following one

$$\partial_t \|u_z\|_2^2 + \|\nabla u_z\|_2^2 \leq C(\|u\|_{H^1} + \|u\|_{H^1}^2) \|u_z\|_2^2 + \frac{1}{2} \|\nabla_H p\|_{L^2(G)}^2 + 2C_2 \|\tilde{u}\|_{L^2(G)}^2 + C(\|u\|_{H^1}^2 + \|f\|_2^2).$$

Step 3: Equation (4.4) implies

$$\begin{aligned} \frac{1}{4} \partial_t \|\tilde{u}\|_4^4 + \frac{1}{2} \|\nabla |\tilde{u}|^2\|_2^2 + \|\tilde{u}\|_{L^2(G)} \|\nabla \tilde{u}\|_2^2 &= - \int_\Omega (\tilde{u} \cdot \nabla_H \bar{u}) \cdot |\tilde{u}|^2 \tilde{u} + \frac{1}{h} \int_\Omega \int_{-h}^0 (\tilde{u} \cdot \nabla_H \tilde{u} + \operatorname{div}_H u \tilde{u}) \, dz \cdot |\tilde{u}|^2 \tilde{u} \\ &\quad + \frac{1}{h} \int_\Omega u_z|_{\Gamma_b} \cdot |\tilde{u}|^2 \tilde{u} + \int_\Omega \tilde{f} \cdot |\tilde{u}|^2 \tilde{u}. \end{aligned}$$

The last term on the right-hand side is bounded by $\|\tilde{f}\|_2 \|\tilde{u}\|_2^3$. We see moreover that

$$\begin{aligned} \|\tilde{u}\|_2^3 &= \|\tilde{u}\|_3^{3/2} \leq C \|\tilde{u}\|_{W^{1/2,2}(\Omega)}^{3/2} \leq C \|\tilde{u}\|_2^{3/4} \left(\|\tilde{u}\|_2 + \|\nabla |\tilde{u}|^2\|_2 \right)^{3/4} \\ &= C \|\tilde{u}\|_4^{3/2} \left(\|\tilde{u}\|_2 + \|\nabla |\tilde{u}|^2\|_2 \right)^{3/4} \leq C \|\tilde{u}\|_4^3 + C \|\tilde{u}\|_4^{3/2} \|\nabla |\tilde{u}|^2\|_2^{3/4}, \end{aligned}$$

where we have used the embedding $W^{1/2,2}(\Omega) \hookrightarrow L^3(\Omega)$ and an interpolation inequality. It then follows that

$$\begin{aligned}
\int_{\Omega} \tilde{f} \cdot |\tilde{u}|^2 \tilde{u} &\leq C \|\tilde{f}\|_2 \|\tilde{u}\|_4^3 + C \|\tilde{f}\|_2 \|\tilde{u}\|_4^{3/2} \|\nabla |\tilde{u}|^2\|_2^{3/4} \\
&= C (\|\tilde{f}\|_2^2 \|\tilde{u}\|_4^4)^{1/2} (\|\tilde{u}\|_4^2)^{1/2} + C \|\tilde{f}\|_2^{2/8} (\|\tilde{f}\|_2^2 \|\tilde{u}\|_4^4)^{3/8} (\|\nabla |\tilde{u}|^2\|_2^2)^{3/8} \\
&\leq \frac{1}{6} \|\nabla |\tilde{u}|^2\|_2^2 + C \|\tilde{f}\|_2^2 \|\tilde{u}\|_4^4 + C \|\tilde{u}\|_4^2 + C \|\tilde{f}\|_2 \\
&\leq \frac{1}{6} \|\nabla |\tilde{u}|^2\|_2^2 + C \|f\|_2^2 \|\tilde{u}\|_4^4 + C \|u\|_{H^1}^2 + C \|f\|_2.
\end{aligned}$$

Therefore, estimate (6.9) in [5] is now replaced by

$$\frac{C_3}{4} \partial_t \|\tilde{u}\|_4^4 + C_3 \|\tilde{u}\|_4 \|\nabla \tilde{u}\|_2^2 \leq C (\|u\|_{H^1}^{2/3} + \|u\|_{H^1} + \|u\|_{H^1}^2 + \|f\|_2^2) \|\tilde{u}\|_4^4 + \frac{1}{4} \|\nabla u_z\|_2^2 + C \|u\|_{H^1}^2 + C \|f\|_2,$$

where $C_3 := 2(C_1 + 2C_2)$.

Step 4: Combining *Steps 1–3*, we see that estimate (6.10) in [5] is now being replaced by

$$\begin{aligned}
\partial_t (8 \|\nabla_H \bar{u}\|_{L^2(G)}^2 + \|u_z\|_2^2 + \frac{C_3}{4} \|\tilde{u}\|_4^4) + \frac{1}{2} (\|\nabla_H p\|_{L^2(G)}^2 + \|\nabla u_z\|_2^2 + C_3 \|\tilde{u}\|_4 \|\nabla_H \tilde{u}\|_2^2) \\
\leq K_1(t) (8 \|\nabla_H \bar{u}\|_{L^2(G)}^2 + \|u_z\|_2^2 + (C_3/4) \|\tilde{u}\|_4^4) + K_2(t),
\end{aligned}$$

where, thanks to (4.2), the functions K_1 and K_2 given by

$$K_1(t) := C(1 + \|u\|_2 + \|u\|_2^2)(\|u\|_{H^1}^{2/3} + \|u\|_{H^1} + \|u\|_{H^1}^2 + \|f\|_2^2),$$

$$K_2(t) := C(1 + \|u\|_2^2 + \|u\|_2^4) \|u\|_{H^1}^2 + C(\|f\|_2 + \|f\|_2^2),$$

are integrable on $[0, T]$. We thus conclude by Gronwall's inequality that

$$(4.5) \quad \|\nabla_H \bar{u}(t)\|_{L^2(G)} + \|u_z(t)\|_2 + \|\tilde{u}(t)\|_4 + \int_0^t (\|\nabla_H p\|_{L^2(G)}^2 + \|\nabla u_z\|_2^2 + \|\tilde{u}\|_4 \|\nabla_H \tilde{u}\|_2^2) d\tau$$

must be bounded by some $B_1(\|a\|_{H^1}, \|f\|_{L_T^2(L^2)}, T)$ for all $t \in [0, T]$.

Step 5: Following the estimates in Step 5 of [5], we obtain now

$$\partial_t \|\nabla u\|_2^2 + \|\Delta u\|_2^2 \leq C \|u_z\|_2^2 \|\nabla u_z\|_2^2 \|\nabla u\|_2^2 + C (\|\bar{u}\|_{H^1(G)}^2 + \|\tilde{u}\|_4^4) \|u\|_{H^1}^2 + C \|f\|_2^2,$$

where, thanks to (4.2) and to the estimate for (4.5),

$$C \|u_z\|_2^2 \|\nabla u_z\|_2^2 \quad \text{and} \quad C (\|\bar{u}\|_{H^1(G)}^2 + \|\tilde{u}\|_4^4) \|u\|_{H^1}^2 + C \|f\|_2^2$$

are integrable on $[0, T]$. Thus, Gronwall's inequality yields the desired estimate (4.1). The proof is complete. \square

5. WEAK-STRONG UNIQUENESS AND PROOF OF THE MAIN RESULT

In this final section we give a proof of Theorem 2.4 based on a weak-strong uniqueness argument for the *initial-value* problem.

More precisely, given $f \in L^2(0, T; L^2(\Omega))$, let v be a weak T -periodic solution corresponding to Proposition 3.1. By properties i) and iii) we infer that there exists $t_0 > 0$ such that $v(t_0) \in \mathbb{H}^1(\Omega)$ and

$$(5.1) \quad \|v(t)\|_2^2 + 2 \int_{t_0}^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(t_0)\|_2^2 + 2 \int_{t_0}^t (f(\tau), v(\tau)) d\tau, \quad t \geq t_0,$$

while from ii) we deduce that, for arbitrary $\mathcal{T} > 0$, the following relation

$$(5.2) \quad \int_{t_0}^t \{ (v, \partial_t \varphi) - (\nabla v, \nabla \varphi) - (v \cdot \nabla_H v + w(v) \partial_z v, \varphi) \} = - \int_{t_0}^t (f, \varphi) + (v(t), \varphi(t)) - (v(t_0), \varphi(t_0))$$

holds for all $t \in (t_0, \mathcal{T}]$ and all $\varphi \in C^1([t_0, \mathcal{T}]; \mathbb{H}^1(\Omega)) \cap L^2(0, \mathcal{T}; \mathbb{H}^2(\Omega))$. Note that in (5.2) we have used the identity

$$(5.3) \quad (\mathbf{v} \cdot \nabla_H \phi + w(\mathbf{v}) \partial_z \phi, \mathbf{v}) = -(\mathbf{v} \cdot \nabla_H \mathbf{v} + w(\mathbf{v}) \partial_z \mathbf{v}, \phi), \quad (\mathbf{v}, \phi) \in \mathbb{H}^1(\Omega) \times \mathbb{H}^2(\Omega),$$

which follows by integration by parts.

We now look at our weak time-periodic solution as a weak solution to the *initial-value problem* with initial data $a \equiv v(t_0)$. Since $v(t_0) \in \mathbb{H}^1(\Omega)$ we may use it also as initial value for the *global* strong solution determined in Proposition 4.1. The assertion of Theorem 2.4 follows provided we are able to show that the weak solution coincides with the strong one.

As a matter of fact, such a weak-strong uniqueness result is already known, under slightly different boundary conditions; see [4, Theorem 1.3]. The arguments used there would equally apply to the case at hand. Nevertheless, we shall sketch a proof here. To this end, let u be the velocity field of the strong solution determined in Proposition 4.1 corresponding to f and initial data $v(t_0)$. Clearly, u satisfies

$$(5.4) \quad \int_{t_0}^t \{ (u, \partial_t \varphi) - (\nabla u, \nabla \varphi) + (u \cdot \nabla_H \varphi + w(u) \partial_z \varphi, u) \} = - \int_{t_0}^t (f, \varphi) + (u(t), \varphi(t)) - (v(t_0), \varphi(t_0)),$$

for all $t \in (t_0, \mathcal{T}]$ and all φ . In addition, thanks to the regularity properties of u , we see that

$$(5.5) \quad \|u(t)\|_2^2 + 2 \int_{t_0}^t \|\nabla u(\tau)\|_2^2 d\tau = \|v(t_0)\|_2^2 + 2 \int_{t_0}^t (f(\tau), u(\tau)) d\tau, \quad t \geq t_0.$$

Next, let

$$v_h(t) := \int_0^{\mathcal{T}} j_h(t-s) v(s) ds \quad \text{and} \quad u_h(t) := \int_0^{\mathcal{T}} j_h(t-s) u(s) ds$$

be the (Friedrichs) time-mollifier of v and u , respectively, where $j_h \in C_c^\infty(-h, h)$, $0 < h < \mathcal{T}$, is even and positive with $\int_{\mathbb{R}} j_h(s) ds = 1$. Then, as is well known,

$$(5.6) \quad \begin{aligned} \lim_{h \rightarrow 0} \int_0^{\mathcal{T}} \|v_h(\tau) - v(\tau)\|_{\mathbb{H}^1}^2 d\tau &= 0, & \operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|v_h(t)\|_2 &\leq \operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|v(t)\|_2, \text{ and} \\ \lim_{h \rightarrow 0} \int_0^{\mathcal{T}} \|u_h(\tau) - u(\tau)\|_{H^2}^2 d\tau &= 0, & \operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|u_h(t)\|_{H^1} &\leq \operatorname{ess\,sup}_{t \in [0, \mathcal{T}]} \|u(t)\|_{H^1}. \end{aligned}$$

Moreover,

$$(5.7) \quad \int_{t_0}^t (v, \partial_t u_h) = - \int_{t_0}^t (u, \partial_t v_h),$$

while the weak continuity of v and u implies

$$(5.8) \quad \lim_{h \rightarrow 0} (u(t), v_h(t)) = \lim_{h \rightarrow 0} (u_h(t), v(t)) = \frac{1}{2} (u(t), v(t)), \quad t \geq t_0.$$

Finally, setting $\sigma := v - u$, by integrating by parts we get

$$(5.9) \quad (\sigma \cdot \nabla_H u + w(\sigma) \partial_z u, u) = 0, \quad \sigma \in \mathbb{H}^1(\Omega).$$

We now replace φ in (5.2) by u_h , then use (5.3) with $\mathbf{v} \equiv u$ and $\phi \equiv \varphi$ in (5.4) and replace in the latter φ by v_h . Summing side by side the resulting equations and employing (2.4), (2.5), (5.6), (5.8) and (5.9) we show that

$$(5.10) \quad \int_{t_0}^t \{ -2(\nabla v, \nabla u) - (\sigma \cdot \nabla_H \sigma + w(\sigma) \partial_z \sigma, u) \} d\tau = - \int_{t_0}^t f \cdot (u + v) d\tau - [(v(t), u(t)) - \|v(t_0)\|_2^2],$$

Adding twice (5.10) to (5.1) and (5.5) we infer

$$\|\sigma(t)\|_2^2 + 2 \int_0^t \|\nabla \sigma(\tau)\|_2^2 d\tau \leq 2 \int_0^t \{(\sigma(\tau) \cdot \nabla_H \sigma(\tau) + w(\sigma) \partial_z \sigma(\tau), u(\tau))\} d\tau.$$

In order to estimate the above right-hand side we use (5.3) and (2.4), (2.5) with $v_1 = v_2 \equiv \sigma$ and $v_3 \equiv u$, respectively, and Poincaré's and Young's inequalities to deduce

$$\begin{aligned} \|\sigma(t)\|_2^2 + 2 \int_{t_0}^t \|\nabla \sigma(\tau)\|_2^2 d\tau &\leq C \int_{t_0}^t \{ \|\sigma(\tau)\|_2^{\frac{1}{2}} \|\sigma(\tau)\|_{H^1}^{\frac{3}{2}} (\|\nabla_H u(\tau)\|_2 + \|\partial_z u(\tau)\|_2^{\frac{1}{2}} \|\partial_z u(\tau)\|_{H^1}^{\frac{1}{2}}) \} d\tau \\ &\leq C \int_{t_0}^t g(\tau) \|\sigma(\tau)\|_2^2 d\tau + \int_{t_0}^t \|\nabla \sigma(\tau)\|_2^2 d\tau, \end{aligned}$$

where $g(\tau) := \|\nabla_H u(\tau)\|_2^4 + \|\partial_z u(\tau)\|_2^2 \|\partial_z u(\tau)\|_{H^1}^2$. By the regularity properties of u it follows that $g \in L^1(t_0, t)$ for arbitrary $t > t_0$, so that by Gronwall's lemma we conclude that

$$\sigma \equiv v - u = 0 \text{ a.e. in } \Omega \times (t_0, t) \text{ for all } t > t_0.$$

By the time-periodicity of v this proves the desired regularity for v . The proof of the Theorem 2.4 is complete.

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STRONG TIME-PERIODIC SOLUTIONS TO THE 3D PRIMITIVE EQUATIONS SUBJECT TO ARBITRARY LARGE FORCES **5**

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